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Regularity properties of the solutions to the 3D Maxwell–Landau–Lifshitz system in weighted Sobolev spaces ☆

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Abstract

This paper is concerned with special regularity properties of the solutions to the Maxwell–Landau–Lifshitz (MLL) system describing ferromagnetic medium. Besides the classical results on the boundedness of $\partial_t \mathbf{m}$, $\partial_t \mathbf{E}$ and $\partial_t \mathbf{H}$ in the spaces $L^\infty(I, L^2(\Omega))$ and $L^2(I, W^{1,2}(\Omega))$ we derive also estimates in weighted Sobolev spaces. This kind of estimates can be used to control the Taylor remainder when estimating the error of a numerical scheme.

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Keywords: Micromagnetism; Regularity results; Ferromagnets

1. Introduction

We solve full Maxwell–Landau–Lifshitz (MLL) system [9,12–14]

$$\partial_t \mathbf{m} = -\mathbf{m} \times (\Delta \mathbf{m} + \mathbf{H}) - \alpha \mathbf{m} \times (\mathbf{m} \times (\Delta \mathbf{m} + \mathbf{H})), \tag{1}$$

$$\partial_t \mathbf{E} + \sigma \mathbf{E} - \nabla \times \mathbf{H} = \mathbf{0}, \tag{2}$$

$$\partial_t \mathbf{H} + \nabla \times \mathbf{E} = -\beta \partial_t \mathbf{m}, \tag{3}$$

$$\nabla \cdot \mathbf{H} + \beta \nabla \cdot \mathbf{m} = 0, \tag{4}$$

$$\nabla \cdot \mathbf{E} = 0, \tag{5}$$

on the time interval $I = (0, T)$ and inside the domain Ω. Constants α and σ are of the physical origin, representing the dissipation parameter and conductivity of the medium, β is a scaling constant. From physical point of view it is desirable to assume $\alpha > 0$, $\sigma \geqslant 0$. Symbols ∂_t and ∂_t^i denote the first and ith time derivative, respectively. In practical applications σ is a "nice" function describing the conductivity of the medium. In general it varies in space. Nevertheless, a non-constant σ would not change the mathematical analysis and therefore we can consider it as a constant.

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We are interested in the cases such as electromagnetic wave propagation, antennas and others, when space-periodic functions occur. Our domain can be then considered as a cube $\Omega = (0, d)^3$ with periodic boundary conditions in each space direction, i.e., for $\mathbf{x} \in \Omega \subset \mathbb{R}^3$, and $t \geqslant 0$,

$$\mathbf{m}(\mathbf{x} + D\mathbf{e}_i, t) = \mathbf{m}(\mathbf{x}, t), \quad \mathbf{H}(\mathbf{x} + D\mathbf{e}_i, t) = \mathbf{H}(\mathbf{x}, t),$$
$$\mathbf{E}(\mathbf{x} + D\mathbf{e}_i, t) = \mathbf{E}(\mathbf{x}, t),$$

where $\mathbf{x} + D\mathbf{e}_i = (x_1, \ldots, x_i + D, \ldots, x_3)$, $i = 1, 2, 3$ and $D > 0$.

The initial conditions read as

$$\mathbf{m}(x, 0) = \mathbf{m}_0(x), \quad \mathbf{H}(x, 0) = \mathbf{H}_0(x), \quad \mathbf{E}(x, 0) = \mathbf{E}_0(x), \quad x \in \Omega \subset \mathbb{R}^3.$$

A crucial observation is, that $|\mathbf{m}| = 1$, for almost all $t \in (0, \infty)$ provided that $|\mathbf{m}_0| = 1$, which is a reasonable physical assumption and that the solution to (1)–(5) is sufficiently smooth. This comes from a scalar multiplication of (1) with $\mathbf{m}$. Then Eq. (1) is equivalent to

$$\partial_t \mathbf{m} - \alpha\Delta\mathbf{m} - \alpha|\nabla\mathbf{m}|^2\mathbf{m} + \mathbf{m} \times \Delta\mathbf{m} = -\mathbf{m} \times \mathbf{H} - \alpha\mathbf{m} \times (\mathbf{m} \times \mathbf{H}). \tag{6}$$

The transformation of Eq. (1) to Eq. (6) is a classical approach used for example in [3,9,18].

1.1. Notations

We use symbols $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle_i$ for scalar products in $\mathbb{R}^3$ and $\mathbb{R}^i$, respectively. For scalar product in the space $L^2(\Omega)$ we use symbol $(\cdot, \cdot)$. Denote the norms in the spaces $L^p(\Omega)$, $W^{k,p}(\Omega)$ by $\|\cdot\|_p$, $\|\cdot\|_{W^{k,p}}$.

2. Overview of known results

The setting of the problem can vary in BCs. Some authors consider periodic BCs in space, such as Guo and Su. In [11,12] they proved the global existence of a weak solution for the three-dimensional M-LL system. For the case of the Neumann BCs Guo and Ding in [9] are concerned with the global existence and the partial regularity for the weak solution. We mention also strong numerical analysis of Monk and Vacus [17]. They proved the existence of a new class of Liapunov functions for the continuous problem, and then also for a variational formulation of the continuous problem. The authors also showed a special result on continuous dependence.

The case, when the exchange term $\Delta\mathbf{m}$ in the LL equation is considered to be zero, was studied by Joly and others in [13,14]. They show that if the Cauchy data are smooth, then the solution remains smooth for all time. In [15,16] the authors are interested in the numerical modeling of absorbing ferromagnetic materials. They proposed a numerical scheme which conserves the magnitude of magnetization, but they did not prove any error estimates in time. For more details we refer to [22].

In [19,20] the authors suggested a new numerical scheme conserving the magnitude of magnetization and they also proved error estimates in time. They considered the LL equation in a simplified form considering a demagnetizing and anisotropy field but without an exchange field. More results on this scheme can be found also in [6,7,21].

Since the M-LL system is a coupled system of the LL equation and Maxwell's equation, we can expect, that the regularity results of the solutions to the M-LL system can copy those obtained for the single LL equation. We mention work of Visintin [23], Alouges and Soyeur [1], Guo and Hong [10]. All these authors studied the existence of a global weak solution for the LL equation.

For overview of computational micromagnetism also with combination with Maxwell's equations see the monograph [18] written by Prohl.

Carbou and Fabrie in [3] proved the local existence and uniqueness of regular solutions to the LL equation in 3D. So in three dimensions, solutions to the LL equation can blow up in a finite time.

The same results can be obtained also for the M-LL system. We study this case in [4]. For the exact solution of the MLL system several regularity results are known. In the next theorem we recall our results from [4]. We obtained local existence and uniqueness of a weak solution to the system (1)–(5). For strong solutions the same results are valid.

Theorem 1. *Suppose that the initial conditions satisfy*

$$\nabla \cdot \mathbf{E}_0 = \nabla \cdot (\mathbf{H}_0 + \beta\mathbf{m}_0) = 0$$

and moreover

$$\mathbf{m}_0 \in W^{2,2}(\Omega) \quad \textit{and} \quad \mathbf{H}_0, \mathbf{E}_0 \in W^{1,2}(\Omega).$$

Then there exists a positive $T^* > 0$ *such that on the interval* $(0, T^*)$ *the weak solution of the system* (6), (2)–(5) *exists and is unique. Moreover the solution satisfies*

$$\sup_{t\in I}\{\|\mathbf{m}\|_{W^{2,2}(\Omega)} + \|\mathbf{E}\|_{W^{1,2}(\Omega)} + \|\mathbf{H}\|_{W^{1,2}(\Omega)}\} \leqslant C, \tag{7}$$

$$\int_0^{T^*} \|\mathbf{m}\|^2_{W^{3,2}} \leqslant C. \tag{8}$$

We can directly derive some other useful estimates on the solution

$$\|\nabla\mathbf{m}\|_4 + \|\mathbf{E}\|_4 + \|\mathbf{H}\|_4 \leqslant C, \tag{9}$$

$$\|\mathbf{m}\|_{L^\infty} \leqslant C. \tag{10}$$

The estimate (9) follows directly from (7) and the embedding $W^{1,2}(\Omega) \hookrightarrow L^4(\Omega)$. For (10) we employ the embedding $W^{2,2}(\Omega) \hookrightarrow L^\infty(\Omega)$.

Multiplication of (1) by $\mathbf{m}$ gives directly

$$\langle \partial_t \mathbf{m}, \mathbf{m}\rangle = 0. \tag{11}$$

3. Regularity of MLL system

We would like to derive the regularity results for the 3D case following the results obtained in 2D case in [18]. We approve that the solutions to MLL system belong to specific weighted Sobolev spaces. We would like to prove that

$$\partial_t^2\mathbf{m} \in L^2(I, L^2(\Omega), \kappa(t)),$$
$$\partial_t\mathbf{m} \in L^\infty(I, W^{1,2}(\Omega), \kappa^{1/2}(t)),$$

where $L^p(I, X, f)$ is a weighted Sobolev space with norm defined by

$$|||u|||_p = \int_0^T \|u\|_X^p f(t)\,\mathrm{d}t.$$

In our case $\kappa(t) := \min\{t, 1\}$.

This type of results are very helpful when proving error estimates for numerical schemes solving the system (6), (2)–(5). We state the main results of this paper in the following theorem.

Theorem 2. *Let the assumptions of Theorem* 1 *be fulfilled. Then there exists a positive* $T^* > 0$ *such that on the interval* $(0, T^*)$ *the solution of the system* (6), (2)–(5) *satisfies*

$$\sup_{t\in I}\{\|\partial_t\mathbf{m}\|_2 + \|\partial_t\mathbf{E}\|_2 + \|\partial_t\mathbf{H}\|_2\} + \left(\int_0^{T^*} \|\partial_t\nabla\mathbf{m}(s)\|_2^2\,\mathrm{d}s\right)^{1/2} \leqslant C, \tag{12}$$

$$\left(\int_0^{T^*} \{\|\partial_t^2\mathbf{m}(s)\|_{W^{-1,2}} + \|\partial_t^2\mathbf{H}(s)\|_{W^{-1,2}} + \|\partial_t^2\mathbf{E}(s)\|_{W^{-1,2}}\}^2\,\mathrm{d}s\right)^{1/2} \leqslant C, \tag{13}$$

$$\sup_{t\in I}\sqrt{\kappa}\|\partial_t\nabla\mathbf{m}\|_2 + \left(\int_0^{T^*} \kappa\{\|\partial_t^2\mathbf{m}(s)\|_2 + \|\partial_t\Delta\mathbf{m}(s)\|_2 + \|\partial_t^2\mathbf{H}(s)\|_2 + \|\partial_t^2\mathbf{E}(s)\|_2\}^2\,\mathrm{d}s\right)^{1/2} \leqslant C, \tag{14}$$

where $\kappa(s) := \min\{s, 1\}$.

Before we prove the previous theorem, let us summarize some integral inequalities derived from the extended Sobolev inequalities, for more details see [8]. For a bounded domain Ω with $\partial\Omega$ in C^2 we have

$$\|u\|_{L^4} \leqslant C\|u\|^{3/4}_{W^{1,2}}\|u\|^{1/4}_{L^2} \leqslant C\|u\|_{L^2} + C\|\nabla u\|^{3/4}_{L^2}\|u\|^{1/4}_{L^2}, \tag{15}$$

$$\|\nabla u\|_{L^4} \leqslant C\|\nabla u\|^{3/4}_{W^{1,2}}\|\nabla u\|^{1/4}_{L^2} \leqslant C\|\nabla u\|_{L^2} + C\|\Delta u\|^{3/4}_{L^2}\|\nabla u\|^{1/4}_{L^2}, \tag{16}$$

for any function u regular enough. The previous estimates can be of course used also in the case of vector valued function $\mathbf{u}$.

Proof of (12), Theorem 2. Time derivation of (6) gives

$$\begin{aligned}\partial_t^2\mathbf{m} = & -\partial_t\mathbf{m}\times(\Delta\mathbf{m}+\mathbf{H}) - \mathbf{m}\times(\partial_t\Delta\mathbf{m}+\partial_t\mathbf{H}) + \alpha\partial_t\Delta\mathbf{m}\\ & + \alpha|\nabla\mathbf{m}|^2\partial_t\mathbf{m} + 2\alpha\langle\nabla\mathbf{m},\partial_t\nabla\mathbf{m}\rangle_9\mathbf{m} + \alpha\partial_t\mathbf{H}\\ & - \alpha\langle\partial_t\mathbf{m},\mathbf{H}\rangle_3\mathbf{m} - \alpha\langle\mathbf{m},\partial_t\mathbf{H}\rangle_3\mathbf{m} - \alpha\langle\mathbf{m},\mathbf{H}\rangle_3\partial_t\mathbf{m}.\end{aligned} \tag{17}$$

Since (11) we get rid of some terms by multiplication of (17) with $\partial_t\mathbf{m}$. Then we integrate the equation over Ω, we provide the integration by parts in the term $(\partial_t\Delta\mathbf{m},\partial_t\mathbf{m})$ incorporating periodic boundary conditions, and thus

$$\begin{aligned}\tfrac{1}{2}\partial_t\|\partial_t\mathbf{m}\|_2^2 + \alpha\|\partial_t\mathbf{m}\|_2^2 = & -(\mathbf{m}\times\partial_t\Delta\mathbf{m},\partial_t\mathbf{m}) - (\mathbf{m}\times\partial_t\mathbf{H},\partial_t\mathbf{m})\\ & + \alpha(\partial_t\mathbf{H},\partial_t\mathbf{m}) + \alpha(|\nabla\mathbf{m}|^2\partial_t\mathbf{m},\partial_t\mathbf{m})\\ & - \alpha(\langle\mathbf{m},\mathbf{H}\rangle_3\partial_t\mathbf{m},\partial_t\mathbf{m}),\end{aligned}$$

using the identity $\langle\partial_t\mathbf{m}\times\cdot,\partial_t\mathbf{m}\rangle = 0$. Performing integration by parts in the first term and using classical integral inequalities we get

$$\begin{aligned}\tfrac{1}{2}\partial_t\|\partial_t\mathbf{m}\|_2^2 + \alpha\|\partial_t\nabla\mathbf{m}\|_2^2 \leqslant & |(\partial_t\mathbf{m}\times\nabla\mathbf{m},\partial_t\nabla\mathbf{m})| + |(\mathbf{m}\times\partial_t\mathbf{H},\partial_t\mathbf{m})|\\ & + \alpha[|(\partial_t\mathbf{H},\partial_t\mathbf{m})| + |(|\nabla\mathbf{m}|^2\partial_t\mathbf{m},\partial_t\mathbf{m})|\\ & + |(\langle\mathbf{m},\mathbf{H}\rangle_3\partial_t\mathbf{m},\partial_t\mathbf{m})|]\\ \leqslant & \|\partial_t\mathbf{m}\|_4\|\nabla\mathbf{m}\|_4\|\partial_t\nabla\mathbf{m}\|_2\\ & + \|\mathbf{m}\|_\infty\|\partial_t\mathbf{H}\|_2\|\partial_t\mathbf{m}\|_2\\ & + \alpha[\|\partial_t\mathbf{H}\|_2\|\partial_t\mathbf{m}\|_2 + \|\nabla\mathbf{m}\|_4^2\|\partial_t\mathbf{m}\|_4^2\\ & + \|\mathbf{m}\|_4\|\mathbf{H}\|_4\|\partial_t\mathbf{m}\|_4^2].\end{aligned}$$

Using (9) and (10) we estimate some terms by a generic constant C. On the term $\|\partial_t\mathbf{m}\|_4\|\partial_t\nabla\mathbf{m}\|_2$ we demonstrate a technique, which will be used very often in next proofs. Applying (15) we get

$$\|\partial_t\mathbf{m}\|_4\|\partial_t\nabla\mathbf{m}\|_2 \leqslant C(\|\partial_t\mathbf{m}\|_2 + \|\partial_t\mathbf{m}\|_2^{1/4}\|\partial_t\nabla\mathbf{m}\|_2^{3/4})\|\partial_t\nabla\mathbf{m}\|_2,$$

which after using two times the Young inequality, first with exponents equal to 2, second with exponents $\frac{8}{7}$ and 8, we conclude

$$\|\partial_t\mathbf{m}\|_4\|\partial_t\nabla\mathbf{m}\|_2 \leqslant C\|\partial_t\mathbf{m}\|_2^2 + \varepsilon\|\partial_t\mathbf{m}\|_2^2.$$

Here ε is a “small” constant. Now, we continue using the previous technique and incorporating the Young inequality to get

$$\tfrac{1}{2}\partial_t\|\partial_t\mathbf{m}\|_2^2 + \alpha\|\partial_t\nabla\mathbf{m}\|_2^2 \leqslant C + C\|\partial_t\mathbf{m}\|_2^2 + \varepsilon\|\partial_t\nabla\mathbf{m}\|_2^2 + C\|\partial_t\mathbf{H}\|_2^2.$$

From (3) and (7) we have directly $\|\partial_t\mathbf{H}\|_2 \leqslant C + C\|\partial_t\mathbf{m}\|_2$ which gives

$$\partial_t\|\partial_t\mathbf{m}\|_2^2 + \|\partial_t\nabla\mathbf{m}\|_2^2 \leqslant C + C\|\partial_t\mathbf{m}\|_2^2 + \varepsilon\|\partial_t\nabla\mathbf{m}\|_2^2 \leqslant C + C\|\partial_t\mathbf{m}\|_2^2$$

setting ε small enough. The use of Gronwall’s lemma confirms one part of (12). Having the estimate for $\|\partial_t\mathbf{m}\|_2$ it is easy to verify the same estimates also for $\|\partial_t\mathbf{H}\|_2$ and $\|\partial_t\mathbf{E}\|_2$, which completes the proof of (12). $\square$

Proof of (13), Theorem 2. We make use of (17)

$$\int_0^T \|\partial_t^2 \mathbf{m}\|_{W^{-1,2}}^2 = \|\partial_t^2 \mathbf{m}\|_{L^2(I,W^{-1,2})}$$
$$\leqslant \sup_{\substack{\boldsymbol{\varphi}\in L^2(I,W^{1,2}),\\ \|\boldsymbol{\varphi}\|_{L^2(I,W^{1,2})}\leqslant 1}} \int_0^T |(\alpha\partial_t \nabla \mathbf{m}, \nabla\boldsymbol{\varphi}) + A_1 + \cdots + A_4 + B_1 + \cdots + B_5|\,\mathrm{d}s.$$

The terms $B_1, \ldots, B_5$ are those ones from (17), that include the magnetic field $\mathbf{H}$. Letters $A_1, \ldots, A_4$ denote the remaining terms. In the analysis of the single LL equation in [5] we derived the same kind of estimates and thus we recall the results

$$|A_1| + \cdots + |A_4| \leqslant C.$$

For the term including $(\alpha\nabla\mathbf{m}_t, \nabla\boldsymbol{\varphi})$ we simply apply (12) to conclude its boundedness.

The terms $B_1, \ldots, B_5$ can be bounded separately. Using (9), (15) and the Young inequality for the term B_1, we get

$$\left|\int_0^T B_1\right| \leqslant C\int_0^T |(\partial_t \mathbf{m} \times \mathbf{H}, \boldsymbol{\varphi})| \leqslant C\int_0^T \|\partial_t \mathbf{m}\|_2 \|\mathbf{H}\|_4 \|\boldsymbol{\varphi}\|_4 \leqslant C\int_0^T \|\partial_t \mathbf{m}\|_2^2 + \|\boldsymbol{\varphi}\|_{W^{1,2}}^2.$$

Finally, using (12) we get the boundedness of B_1. Similar procedures can be done also for other terms $B_2, \ldots, B_5$. Since there arises no problematic term we do not go into details and we conclude that

$$\int_0^T \|\partial_t^2 \mathbf{m}\|_{W^{-1,2}}^2 \leqslant C.$$

To prove the rest of (13) we incorporate space derivation of Maxwell's equation (3) to get

$$\int_0^{T^*} \|\partial_t \nabla\times\mathbf{H}\|_{W^{-1,2}}^2 \leqslant \int_0^{T^*} \|\nabla\times\nabla\times\mathbf{E}\|_{W^{-1,2}}^2 + \beta\int_0^{T^*} \|\partial_t \nabla\mathbf{m}\|_{W^{-1,2}}^2$$
$$\leqslant \int_0^{T^*} \|\nabla\times\mathbf{E}\|_2^2 + \beta\int_0^{T^*} \|\partial_t \mathbf{m}\|_2^2 \leqslant C,$$

after using (12). Now, we derive (2) in time and employ the previous result to obtain

$$\int_0^{T^*} \|\partial_t^2 \mathbf{E}\|_{W^{-1,2}}^2 \leqslant \sigma\int_0^{T^*} \|\partial_t \mathbf{E}\|_{W^{-1,2}}^2 + \int_0^{T^*} \|\partial_t \nabla\times\mathbf{H}\|_{W^{-1,2}}^2$$
$$\leqslant \sigma\int_0^{T^*} \|\partial_t \mathbf{E}\|_2^2 + C \leqslant C$$

which finally concludes the proof of (13). □

Proof of (14), Theorem 2. We multiply (17) by $-\partial_t \Delta\mathbf{m}$ and integrate over Ω. It is only formal step, but it can be done rigorous by different quotient method. Because of periodic boundary conditions we get

$$\tfrac{1}{2}\partial_t \|\partial_t \nabla\mathbf{m}\|_2^2 + \alpha\|\partial_t \Delta\mathbf{m}\|_2^2 \leqslant C_1 + \cdots + C_4 + D_1 + \cdots + D_4. \tag{18}$$

Again, the terms including the magnetic field $\mathbf{H}$ are denoted by $D_1, \ldots, D_4$ and the other terms are denoted by $C_1, \ldots, C_4$. We recall the work that we have done in [5], where we have studied the case without Maxwell's equations. We have obtained

$$C_1 + \cdots + C_4 \leqslant C_\varepsilon \|\partial_t \mathbf{m}\|_2^2 + C_\varepsilon \|\partial_t \nabla\mathbf{m}\|_2^2 + \varepsilon\|\partial_t \Delta\mathbf{m}\|_2^2,$$

where ε can be chosen arbitrarily small. We deal with terms $D_1, \ldots, D_4$ independently. First we have

$$D_1 := |(\partial_t \mathbf{m}\times\mathbf{H} + \mathbf{m}\times\partial_t \mathbf{H}, -\partial_t \Delta\mathbf{m})|$$
$$\leqslant \|\partial_t \mathbf{m}\|_4 \|\mathbf{H}\|_4 \|\partial_t \Delta\mathbf{m}\|_2 + \|\mathbf{m}\|_\infty \|\partial_t \mathbf{H}\|_2 \|\partial_t \Delta\mathbf{m}\|_2.$$

Using (9), (16), and the Young inequalities with appropriate exponents, we get

$$D_1 \leqslant C_\varepsilon \|\partial_t \mathbf{m}\|_2^2 + C_\varepsilon \|\partial_t \nabla \mathbf{m}\|_2^2 + \varepsilon \|\partial_t \Delta \mathbf{m}\|_2^2 + C_\varepsilon \|\partial_t \mathbf{H}\|_2^2.$$

Following this pattern the estimation of terms $D_2, \dots, D_5$ becomes straightforward. Consequently, we obtain

$$\begin{aligned} D_2 + \cdots + D_4 = \alpha[&|(\langle \partial_t \mathbf{m}, \mathbf{H}\rangle + \langle \mathbf{m}, \partial_t \mathbf{H}\rangle, \langle \mathbf{m}, -\partial_t \Delta \mathbf{m}\rangle)| \\ &+ |(\langle \mathbf{m}, \mathbf{H}\rangle, \langle \partial_t \mathbf{m}, -\partial_t \Delta \mathbf{m}\rangle)| + |(\partial_t \mathbf{H}, -\partial_t \Delta \mathbf{m})|] \\ &\leqslant C_\varepsilon \|\partial_t \mathbf{m}\|_2^2 + C_\varepsilon \|\partial_t \nabla \mathbf{m}\|_2^2 + \varepsilon \|\partial_t \Delta \mathbf{m}\|_2^2 + C_\varepsilon \|\partial_t \mathbf{H}\|_2^2. \end{aligned}$$

Setting ε small enough and using (12) we continue with (18)

$$\partial_t \|\partial_t \nabla \mathbf{m}\|_2^2 + \|\partial_t \Delta \mathbf{m}\|_2^2 \leqslant C + C\|\partial_t \nabla \mathbf{m}\|_2^2.$$

Multiplying by the time weight $\kappa(s) = \min\{1, s\}$ leads to

$$\kappa(s)\partial_t \|\partial_t \nabla \mathbf{m}\|_2^2 + \kappa(s)\|\partial_t \Delta \mathbf{m}\|_2^2 \leqslant C\kappa(s) + C\kappa(s)\|\partial_t \nabla \mathbf{m}\|_2^2.$$

We integrate both sides of the previous equation, integrating by parts in the left term to get

$$\begin{aligned} &[\kappa(s)\|\partial_t \nabla \mathbf{m}(s)\|_2^2]_0^t - \int_0^t (\kappa'(s))\|\partial_t \nabla \mathbf{m}\|_2^2 + \int_0^t \kappa(s)\|\partial_t \Delta \mathbf{m}\|_2^2 \\ &\quad \leqslant C\int_0^t \kappa(s) + C\int_0^t \kappa(s)\|\partial_t \nabla \mathbf{m}\|_2^2 \\ &\kappa(t)\|\partial_t \nabla \mathbf{m}(t)\|_2^2 + \int_0^t \kappa(s)\|\partial_t \Delta \mathbf{m}\|_2^2 \\ &\quad \leqslant \int_0^t \|\partial_t \nabla \mathbf{m}\|_2^2 + C + C\int_0^t \kappa(s)\|\partial_t \nabla \mathbf{m}\|_2^2, \end{aligned}$$

because $\kappa'(s) \leqslant 1$. Now we use (12) and we arrive at the formula

$$\kappa(t)\|\partial_t \nabla \mathbf{m}(t)\|_2^2 + \int_0^t \kappa(s)\|\partial_t \Delta \mathbf{m}\|_2^2 \leqslant C_\alpha + C_\alpha \int_0^t \kappa(s)\|\partial_t \nabla \mathbf{m}\|_2^2.$$

We are ready to use Gronwall's lemma to obtain

$$\sqrt{\kappa}\|\partial_t \nabla \mathbf{m}\|_2 + \left(\int_0^T \kappa\|\partial_t \Delta \mathbf{m}\|_2^2\right)^{1/2} \leqslant C. \tag{19}$$

Next we multiply (17) with $\partial_t^2 \mathbf{m}$ to get

$$\|\partial_t^2 \mathbf{m}\|_2^2 + \frac{\alpha}{2}\partial_t \|\partial_t \nabla \mathbf{m}\|_2^2 \leqslant C(E_1 + \cdots + E_4 + F_1 + \cdots + F_5). \tag{20}$$

As before, by $F_1, \dots, F_5$ we denote terms including the magnetic field $\mathbf{H}$ and by $E_1, \dots, E_4$ we denote the other terms. The boundary terms vanish. From [5] we get directly

$$E_1 + \cdots + E_4 \leqslant C_\varepsilon \|\partial_t \mathbf{m}\|_2^2 + C_\varepsilon \|\partial_t \nabla \mathbf{m}\|_2^2 + C_\varepsilon \|\partial_t \Delta \mathbf{m}\|_2^2 + \varepsilon \|\mathbf{m}_{tt}\|_2^2.$$

Further, we continue with the estimation of F_1

$$\begin{aligned} F_1 &:= |(\partial_t \mathbf{m} \times \mathbf{H} + \mathbf{m} \times \partial_t \mathbf{H}, \partial_t^2 \mathbf{m})| \\ &\leqslant \|\partial_t \mathbf{m}\|_4 \|\mathbf{H}\|_4 \|\partial_t^2 \mathbf{m}\|_2 + \|\mathbf{m}\|_\infty \|\partial_t \mathbf{H}\|_2 \|\partial_t^2 \mathbf{m}\|_2. \end{aligned}$$

Using (9), (16), and the Young inequalities with appropriate exponents, we get

$$F_1 \leqslant C_\varepsilon \|\partial_t \mathbf{m}\|_2^2 + C_\varepsilon \|\partial_t \nabla \mathbf{m}\|_2^2 + \varepsilon \|\partial_t^2 \mathbf{m}\|_2^2 + C_\varepsilon \|\partial_t \mathbf{H}\|_2^2.$$

Similarly as before, the estimation of terms $F_2, \dots, F_5$ is straightforward. We obtain

$$\begin{aligned} F_2 + \cdots + F_4 = \alpha[&|(\langle\partial_t \mathbf{m}, \mathbf{H}\rangle + \langle\mathbf{m}, \partial_t \mathbf{H}\rangle, \langle\mathbf{m}, \partial_t^2 \mathbf{m}\rangle)| \\ &+ |(\langle\mathbf{m}, \mathbf{H}\rangle, \langle\partial_t \mathbf{m}, \partial_t^2 \mathbf{m}\rangle)| + |(\partial_t \mathbf{H}, \partial_t^2 \mathbf{m})|] \\ &\leqslant C_\varepsilon \|\partial_t \mathbf{m}\|_2^2 + C_\varepsilon \|\partial_t \nabla \mathbf{m}\|_2^2 + \varepsilon \|\partial_t^2 \mathbf{m}\|_2^2 + C_\varepsilon \|\partial_t \mathbf{H}\|_2^2. \end{aligned}$$

Now, using the previous estimates and (12) we can continue in (20) setting ε small enough to arrive at

$$\|\mathbf{m}_{tt}\|_2^2 + \partial_t \|\partial_t \nabla \mathbf{m}\|_2^2 \leqslant C + C\|\partial_t \Delta \mathbf{m}\|_2^2 + C\|\partial_t \nabla \mathbf{m}\|_2^2.$$

Multiplication by $\kappa(s)$ followed by time integration now leads to

$$\int_0^t \kappa(s)\|\mathbf{m}_{tt}\|_2^2 + \kappa(t)\|\partial_t \nabla \mathbf{m}\|_2^2 \leqslant C\int_0^t \kappa(t) + C\int_0^t \kappa(s)\|\partial_t \Delta \mathbf{m}\|_2^2 + C\int_0^t \kappa(s)\|\partial_t \nabla \mathbf{m}\|_2^2.$$

The use of (19) leads us to the desired result

$$\int_0^t \kappa(s)\|\mathbf{m}_{tt}\|^2 + \kappa(t)\|\partial_t \nabla \mathbf{m}\|_2^2 \leqslant C. \tag{21}$$

To finish the proof of (14) we employ the Maxwell's equations. Summation of time derivative of (2) and space derivative of (3) gives

$$\partial_t^2 \mathbf{E} + \sigma \partial_t \mathbf{E} + \nabla \times \nabla \times \mathbf{E} = -\beta \partial_t \nabla \times \mathbf{m}.$$

Multiplication by $\partial_t^2 \mathbf{E}$ leads to

$$\|\partial_t^2 \mathbf{E}\|_2^2 + \tfrac{1}{2}\partial_t \|\nabla \times \mathbf{E}\|_2^2 \leqslant \varepsilon \|\partial_t^2 \mathbf{E}\|_2^2 + C_\varepsilon \sigma \|\partial_t \mathbf{E}\|_2^2 + \beta C_\varepsilon \|\partial_t \nabla \mathbf{m}\|_2^2.$$

After considering ε small enough and using (12) we obtain

$$\|\partial_t^2 \mathbf{E}\|_2^2 + \partial_t \|\nabla \times \mathbf{E}\|_2^2 \leqslant C + C\|\partial_t \nabla \mathbf{m}\|_2^2.$$

Next, we multiply the previous relation by κ and integrate over Ω to arrive at

$$\int_0^{T^*} \kappa \|\partial_t^2 \mathbf{E}(s)\|_2^2 \, \mathrm{d}s + \|\nabla \times \mathbf{E}(T^*)\|_2^2 \leqslant C + C\int_0^{T^*} \kappa \|\partial_t \nabla \mathbf{m}\|_2^2 \leqslant C,$$

where we have used (21) at the end.

This verifies upper bound for $\|\sqrt{\kappa}\partial_t^2 \mathbf{E}\|_{L^2(I, L^2(\Omega))}$.

Summation of time derivative of (3) and space derivative of (2) gives

$$\partial_t^2 \mathbf{H} - \sigma \nabla \times \mathbf{E} + \nabla \times \nabla \times \mathbf{H} = -\beta \partial_t^2 \mathbf{m}.$$

Multiplication by $\partial_t^2 \mathbf{H}$ leads to

$$\|\partial_t^2 \mathbf{H}\|_2^2 + \tfrac{1}{2}\partial_t \|\nabla \times \mathbf{H}\|_2^2 \leqslant \varepsilon \|\partial_t^2 \mathbf{H}\|_2^2 + C_\varepsilon \sigma \|\nabla \times \mathbf{E}\|_2^2 + \beta C_\varepsilon \|\partial_t^2 \mathbf{m}\|_2^2.$$

We consider ε small enough and using (12) we arrive at

$$\|\partial_t^2 \mathbf{H}\|_2^2 + \partial_t \|\nabla \times \mathbf{H}\|_2^2 \leqslant C + C\|\partial_t^2 \mathbf{m}\|_2^2.$$

Next, we multiply the previous relation by κ and integrate over Ω to obtain

$$\int_0^{T^*} \kappa \|\partial_t^2 \mathbf{H}(s)\|_2^2 \, \mathrm{d}s + \|\nabla \times \mathbf{H}(T^*)\|_2^2 \leqslant C + C\int_0^{T^*} \kappa \|\partial_t^2 \mathbf{m}\|_2^2 \leqslant C,$$

where we have used (21) at the end.

This verifies upper bound also for $\|\sqrt{\kappa}\partial_t^2 \mathbf{H}\|_{L^2(I, L^2(\Omega))}$. This completes the proof of inequality (14). □

4. Conclusions

We have obtained regularity results for MLL system describing micromagnetic phenomena. These results are essential for rigorous analysis of many numerical schemes. The main advantage lies in the fact that we obtained bounds for second time derivatives even if we did not assumed a priori any estimates on the time derivatives in time 0.

We refer also to [2] for the overview of numerical schemes dealing with single LL equation or full MLL system.

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